Polynomial Heisenberg algebras

2004 J. Phys. A: Math. Gen. 37 10349

(http://iopscience.iop.org/0305-4470/37/43/022)

Institute of Physics Publishing | Journal of Physics A: Mathematical and General
J. Phys. A: Math. Gen. **37** (2004) 10349–10362 | PII: S0305-4470(04)74688-0

Polynomial Heisenberg algebras

Juan M Carballo[1], **David J Fernández C**[2], **Javier Negro**[3] **and Luis M Nieto**[3,4]

[1] ESFM and ESIA, Instituto Politécnico Nacional, U P Adolfo López Mateos, 07738 México DF, Mexico
[2] Departamento de Física, CINVESTAV, AP 14-740, 07000 México DF, Mexico
[3] Departamento de Física Teórica, Universidad de Valladolid, 47005 Valladolid, Spain

Received 15 January 2004
Published 14 October 2004
Online at stacks.iop.org/JPhysA/37/10349
doi:10.1088/0305-4470/37/43/022

Abstract
Polynomial deformations of the Heisenberg algebra are studied in detail. Some of their natural realizations are given by the higher order susy partners (and not only by those of first order, as is already known) of the harmonic oscillator for even-order polynomials. Here, it is shown that the susy partners of the radial oscillator play a similar role when the order of the polynomial is odd. Moreover, it will be proved that the general systems ruled by such kinds of algebras, in the quadratic and cubic cases, involve Painlevé transcendents of types IV and V, respectively.

PACS numbers: 11.30.Pb, 03.65.Ge, 03.65.Fd, 02.30.Gp

1. Introduction

Deformations of the standard Lie algebras play an important role in diverse interesting problems in physics. We can mention just a couple of examples: the Higgs algebra [1] and the applications to some exactly solvable Hamiltonians [2]. In these structures, some of the commutators, which in the Lie case are linear combinations of the generators, are replaced by certain non-linear functions [3, 4]. If the Lie algebra is associated with an initial Hamiltonian, the deformed Lie algebra will lead to another Hamiltonian whose spectrum will be a certain variant of the original one.

In this paper we are interested in polynomial Heisenberg algebras, i.e., we will study systems for which the commutators of the Hamiltonian H with the annihilation and creation (ladder) operators $L^{\pm}$ are the same as for the harmonic oscillator, but the commutator of L^- and L^+ is an mth-order polynomial $P_m(H)$ in H. Concrete realizations of these polynomial algebras are built by taking $L^{\pm}$ as $(m+1)$th differential operators [5–12]. For example, the higher order supersymmetric (hsusy) partners of the harmonic oscillator provide such

[4] Author to whom any correspondence should be addressed.

realizations for even values of m [11, 12]. Here, we will show that the hsusy partners of the radial oscillator do the same when m is odd.

Another important question is to study, not just particular realizations, but the characterization of the most general systems ruled by polynomial Heisenberg algebras. It will be seen that the difficulty involved in this problem grows with increasing order of the polynomial: for $m = 0$ and $m = 1$ (linear case) these systems are precisely the harmonic and radial oscillators, respectively [5–7, 9]. We will analyse the next step by showing that, for $m = 2$ and $m = 3$, the determination of the potentials requires us to solve Painlevé equations of types IV and V, respectively [7, 8, 13]. Further steps, in which the presence of much more complex differential equations is obvious, are out of the scope of the present work. It is worth mentioning that, by reading back the results for the susy partners of the harmonic and radial oscillators, explicit solutions of these Painlevé equations can be immediately supplied, a simple fact which is not well known in the mathematical literature.

The paper is organized as follows. In section 2 we discuss the polynomial deformations of the Heisenberg algebra, in particular, the possible spectra which can be found. In section 3 the higher order supersymmetric quantum mechanics (hsusy QM) will be introduced, and the corresponding susy partners of the harmonic and radial oscillators will be analysed. We will look for the general systems ruled by the polynomial Heisenberg algebras in section 4, where we will realize the growing complexity arising as m is increased. We finish the paper with some conclusions.

2. Polynomial deformations of the Heisenberg algebra

Polynomial Heisenberg algebras of mth order are deformations of the oscillator algebra, where there are two standard commutation relationships

$$[H, L^{\pm}] = \pm L^{\pm} \tag{2.1}$$

and an atypical one characterizing the deformation,

$$[L^{-}, L^{+}] \equiv N(H+1) - N(H) = P_m(H) \tag{2.2}$$

where the generalized number operator is $N(H) \equiv L^{+}L^{-}$. The corresponding systems are described by the Schrödinger Hamiltonian

$$H = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) \tag{2.3}$$

where $L^{\pm}$ are $(m+1)$th-order differential ladder operators, $N(H)$ is a polynomial in H factorized as

$$N(H) = \prod_{i=1}^{m+1}(H - \mathcal{E}_i) \tag{2.4}$$

and $P_m(H)$ in (2.2) is an mth-order polynomial in H. The algebraic structure generated by $\{H, L^{-}, L^{+}\}$ provides information on the spectrum $\mathrm{Sp}(H)$ of H [6, 11, 12, 14]. Indeed, let us consider the solution space of the $(m+1)$th-order differential equation (the kernel K_{L^-} of L^-):

$$L^{-}\psi = 0 \quad \Rightarrow \quad L^{+}L^{-}\psi = \prod_{i=1}^{m+1}(H - \mathcal{E}_i)\psi = 0. \tag{2.5}$$

As K_{L^-} is invariant under H, it is natural to select as the basis of K_{L^-} these solutions which are simultaneously eigenstates of H with eigenvalues $\mathcal{E}_i$

$$H\psi_{\mathcal{E}_i} = \mathcal{E}_i\psi_{\mathcal{E}_i} \tag{2.6}$$

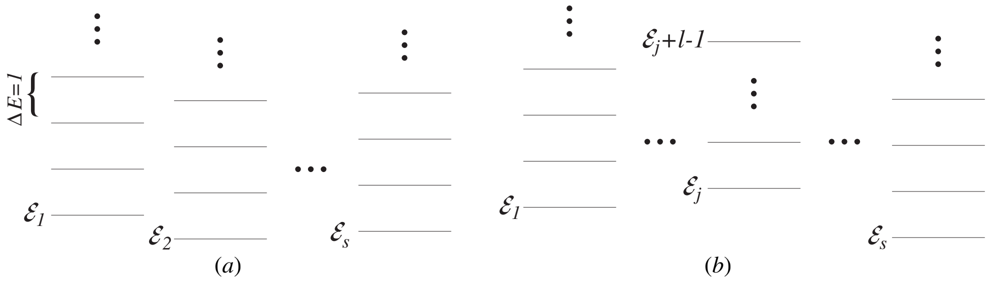

Figure 1. Possible spectra for a Hamiltonian satisfying (2.1)–(2.4) and having s extremal states. In case (a) s infinite ladders are obtained by acting with L^+. In case (b) there are $s-1$ infinite ladders, and a finite ladder (the jth one) is built out of $\psi_{\mathcal{E}_j}$, taking into account (2.7).

becoming the *extremal* states of the $m+1$ *mathematical ladders* of spacing $\Delta E = 1$ starting from $\mathcal{E}_i$. If s of these states are physically meaningful, $\{\psi_{\mathcal{E}_i}, i = 1, \dots, s\}$, then by acting iteratively with L^+, s physical energy ladders can be constructed (see figure 1(a)).

It could happen [11] that for the ladder starting from $\mathcal{E}_j$ there is an $l \in \mathbb{N}$ such that

$$(L^+)^{l-1}\psi_{\mathcal{E}_j} \neq 0 \qquad (L^+)^l \psi_{\mathcal{E}_j} = 0. \tag{2.7}$$

Then, by analysing $L^-(L^+)^l\psi_{\mathcal{E}_j} = 0$ it is seen that another root of (2.4) must be $\mathcal{E}_k = \mathcal{E}_j + l$, $k \in \{s+1, \dots, m+1\}$, $j \in \{1, \dots, s\}$. Hence, $\mathrm{Sp}(H)$ will contain $s-1$ infinite ladders and a finite one of length l, starting from $\mathcal{E}_j$ and ending at $\mathcal{E}_j + l - 1$ (see figure 1(b)).

We conclude that the spectrum of systems described by polynomial Heisenberg algebras of order m can have at most $m+1$ infinite ladders. Note that pairs of ladder operators for the harmonic oscillator system satisfying (2.1)–(2.4), with $m > 0$, can be constructed simply by taking $L^- = aP(H)$, $L^+ = P(H)a^\dagger$, where $a^\dagger$, a are the Heisenberg creation and annihilation operators, and $P(H)$ is a real polynomial in H [4]. In our context these deformations, that we will call reducible, are artificial since for the same system we already have operators $a^\dagger$, a obeying a much simpler algebra. We will be mainly interested in the search for intrinsic, non-reducible, deformed algebras.

3. Higher order supersymmetric quantum mechanics

Let two Schrödinger Hamiltonians H_0, H_k of the form (2.3) be intertwined by differential operators $B^\dagger$, B of kth order [11, 12, 15–19]

$$H_k B^\dagger = B^\dagger H_0 \qquad H_0 B = B H_k \tag{3.1}$$

where $B^\dagger$ is the adjoint of B and the Hamiltonians are assumed to be self-adjoint. The standard supersymmetry algebra

$$[\mathsf{Q}_i, \mathsf{H}_{\mathrm{ss}}] = 0 \qquad \{\mathsf{Q}_i, \mathsf{Q}_j\} = \delta_{ij}\mathsf{H}_{\mathrm{ss}} \qquad i, j = 1, 2 \tag{3.2}$$

is realized by choosing

$$\mathsf{Q} = \begin{pmatrix} 0 & B^\dagger \\ 0 & 0 \end{pmatrix} \qquad \mathsf{Q}^\dagger = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \tag{3.3}$$

$$\mathsf{Q}_1 = \frac{\mathsf{Q}^\dagger + \mathsf{Q}}{\sqrt{2}} \qquad \mathsf{Q}_2 = \frac{\mathsf{Q}^\dagger - \mathsf{Q}}{\mathrm{i}\sqrt{2}} \qquad \mathsf{H}_{\mathrm{ss}} = \begin{pmatrix} B^\dagger B & 0 \\ 0 & BB^\dagger \end{pmatrix}. \tag{3.4}$$

In this so-called kth-order supersymmetric quantum mechanics (k-susy QM) there is a polynomial relationship between H_{ss} and the diagonal matrix H_{d} involving H_0 and H_k:

$$\mathsf{H}_{\mathrm{d}} = \begin{pmatrix} H_k & 0 \\ 0 & H_0 \end{pmatrix} \qquad \mathsf{H}_{\mathrm{ss}} = (\mathsf{H}_{\mathrm{d}} - \epsilon_1) \cdots (\mathsf{H}_{\mathrm{d}} - \epsilon_k). \tag{3.5}$$

The standard susy QM is obtained through the first-order intertwining operator

$$B^\dagger = A_1^\dagger = \frac{1}{\sqrt{2}}\left[-\frac{\mathrm{d}}{\mathrm{d}x} + \alpha_1(x,\epsilon_1)\right] \tag{3.6}$$

which leads to the typical relation between the potentials $V_0(x)$ and$V_1(x)$,

$$V_1(x) = V_0(x) - \alpha_1'(x,\epsilon_1) \tag{3.7}$$

where $\alpha_1(x,\epsilon_1)$ satisfies the Riccati equation:

$$\alpha_1'(x,\epsilon_1) + \alpha_1^2(x,\epsilon_1) = 2[V_0(x) - \epsilon_1]. \tag{3.8}$$

For a given potential $V_0(x)$ and factorization energy ϵ_1, the generation of $V_1(x)$ requires either solving (3.8) or the corresponding Schrödinger equation

$$-\frac{u_1''}{2} + V_0(x)u_1 = \epsilon_1 u_1 \qquad \alpha_1(x,\epsilon_1) = \frac{u_1'}{u_1}. \tag{3.9}$$

On the other hand, if $B^\dagger$ is of order $k > 1$ the potential $V_k(x)$ can be found either through Crum determinants [16] or by defining a sequence of Hamiltonians $H_0, \ldots, H_k$ intertwined by first-order operators $A_i^\dagger = \frac{1}{\sqrt{2}}\big[-\frac{\mathrm{d}}{\mathrm{d}x} + \alpha_i(x,\epsilon_i)\big]$ [17, 18]:

$$H_i A_i^\dagger = A_i^\dagger H_{i-1} \qquad i = 1, \ldots, k. \tag{3.10}$$

Taking into account (3.6)–(3.7), the ith potential reads $V_i(x) = V_{i-1}(x) - \alpha_i'(x,\epsilon_i)$, where

$$\alpha_i'(x,\epsilon_i) + \alpha_i^2(x,\epsilon_i) = 2[V_{i-1}(x) - \epsilon_i]. \tag{3.11}$$

By adopting the identifications

$$B^\dagger = A_k^\dagger \cdots A_1^\dagger \qquad B = A_1 \cdots A_k \tag{3.12}$$

it turns out that the final potential reads

$$V_k(x) = V_0(x) - \sum_{i=1}^{k} \alpha_i'(x,\epsilon_i). \tag{3.13}$$

The corresponding α_i are found through the Bäcklund-type transformation [7, 8, 17, 18]:

$$\alpha_i(x,\epsilon_i) = -\alpha_{i-1}(x,\epsilon_{i-1}) - \frac{2(\epsilon_{i-1} - \epsilon_i)}{\alpha_{i-1}(x,\epsilon_{i-1}) - \alpha_{i-1}(x,\epsilon_i)}. \tag{3.14}$$

Its iterations show that the right-hand side of (3.13) depends just on k solutions $\alpha_1(x,\epsilon_i)$ of the first Riccati equation (3.11) with factorization energies ϵ_i, $i = 1, \ldots, k$.

The hsusy QM is useful to generate solvable potentials from a given initial one. Moreover, we will see next that the hsusy partners of the harmonic and radial oscillators realize in a natural way the polynomial Heisenberg algebras.

3.1. Higher order susy partners of the harmonic oscillator

Let us consider the harmonic oscillator potential $V_0(x) = x^2/2$. The corresponding energy eigenvalues and eigenfunctions are $E_n^{(0)} = n + \frac{1}{2}$, $\psi_n^{(0)}(x) \propto \exp\left(-\frac{x^2}{2}\right)H_n(x)$, $n = 0, 1, \ldots,$

where $H_n(x)$ are Hermite polynomials. The standard ladder operators $a = \frac{1}{\sqrt{2}}\big(\frac{\mathrm{d}}{\mathrm{d}x} + x\big)$, $a^\dagger = \frac{1}{\sqrt{2}}\big(-\frac{\mathrm{d}}{\mathrm{d}x} + x\big)$ connect the $\psi_n^{(0)}$ as follows: $a\psi_n^{(0)} = \sqrt{n}\psi_{n-1}^{(0)}$, $a^\dagger\psi_n^{(0)} = \sqrt{n+1}\psi_{n+1}^{(0)}$. The algebra generated by $\{H_0, a, a^\dagger\}$ is of the type (2.1)–(2.4), with $m = 0$ and $\mathcal{E}_1 = \frac{1}{2}$.

In order to generate the hsusy partner potentials $V_k(x)$ by creating k new levels $\epsilon_1, \ldots, \epsilon_k$ below $E_0^{(0)}$, we need the general solution to the Schrödinger equation (3.9). Up to a constant factor, we have [11]

$$u_1(x) = \exp\left(-\frac{x^2}{2}\right)\left[{}_1F_1\left(\frac{1-2\epsilon_1}{4}, \frac{1}{2}; x^2\right) + 2x\nu_1 \frac{\Gamma\left(\frac{3-2\epsilon_1}{4}\right)}{\Gamma\left(\frac{1-2\epsilon_1}{4}\right)}{}_1F_1\left(\frac{3-2\epsilon_1}{4}, \frac{3}{2}; x^2\right)\right]. \tag{3.15}$$

To avoid singularities for the 1-susy case we must have $|\nu_1| < 1$. The corresponding ν_1 restriction in the higher order situation is, in general, different. The eigenfunctions of H_k are found by applying the $B^\dagger$ of (3.12) to the oscillator eigenfunctions. Thus, the spectrum is $\mathrm{Sp}(H_k) = \big\{\epsilon_i, E_n^{(0)}, i = 1, \ldots, k, n = 0, 1, \ldots\big\}$, a fact that can be explained by means of the polynomial algebra (2.1)–(2.4). Indeed, the natural ladder operators for $H \equiv H_k$ are [11, 20]

$$L^- = B^\dagger a B \qquad L^+ = B^\dagger a^\dagger B \tag{3.16}$$

where B, $B^\dagger$ are the intertwining operators of (3.12). As L^- and L^+ are of $(2k+1)$th order, it turns out that $N(H) = L^+L^-$ is a $(2k+1)$th-order polynomial in H, namely,

$$N(H) = \left(H - \frac{1}{2}\right)\prod_{i=1}^{k}(H - \epsilon_i - 1)(H - \epsilon_i). \tag{3.17}$$

The pairs of roots $\{\epsilon_i, \epsilon_i + 1\}$ in (3.17) imply the existence of k one-step ladders in $\mathrm{Sp}(H)$, one at each ϵ_i. There is also an infinite one starting from 1/2. Since $[L^-, L^+] = P_{2k}(H)$, we see that the even-order polynomial algebras (2.1)–(2.4) are realized naturally by the hsusy partners of the harmonic oscillator [11].

3.2. Higher order susy partners of the radial oscillator

Let us consider now the potential

$$V_0(x) = \frac{x^2}{8} + \frac{l(l+1)}{2x^2} \qquad x > 0 \quad l \geqslant 0. \tag{3.18}$$

Throughout this paper we will refer to this system as the radial oscillator. It is known that its spectrum can be built up algebraically using the following second-order ladder operators A^-, A^+ [19, 21]:

$$A^- = \frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x^2}{4} - \frac{l(l+1)}{x^2} + \frac{1}{2}\right) \tag{3.19}$$

$$A^+ = \frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x^2}{4} - \frac{l(l+1)}{x^2} - \frac{1}{2}\right). \tag{3.20}$$

Two ladders can be constructed out of the two eigenstates of H_0 annihilated by A^-. The physical ladder starts from the extremal state

$$\psi_{\mathcal{E}_1}^{(0)}(x) \propto x^{l+1}\exp\left(-\frac{x^2}{4}\right) \tag{3.21}$$

which is square-integrable in $[0, \infty)$ and vanishes at the extremes of that interval. (The last is the standard requirement for systems with a singularity at $x = 0$ of kind (3.18); we will adopt here this boundary condition in order to characterize the spectrum of H_0.) The associated eigenvalue is $\mathcal{E}_1 = \frac{l}{2} + \frac{3}{4} \equiv E_0^{(0)}$, and the next eigenstates are obtained by acting with powers of A^+ on $\psi_{\mathcal{E}_1}^{(0)} \equiv \psi_0^{(0)}$. The second ladder departs from the other extremal state

$$\psi_{\mathcal{E}_2}^{(0)}(x) \propto x^{-l} \exp\left(-\frac{x^2}{4}\right) \tag{3.22}$$

of eigenvalue $\mathcal{E}_2 = -\frac{l}{2} + \frac{1}{4}$. This state is unphysical because, at $x = 0$, $\psi_{\mathcal{E}_2}^{(0)}(x)$ diverges for $l > 0$ and it does not vanish for $l = 0$. Thus, $\mathrm{Sp}(H_0) = \left\{E_n^{(0)} = n + \frac{l}{2} + \frac{3}{4}, n = 0, 1, \ldots\right\}$.

To implement now the susy techniques, we solve the Schrödinger equation (3.9) with the potential (3.18). Up to a constant factor, the general solution is given by [19]

$$\begin{aligned} u_1(x) = \frac{\exp\left(-\frac{x^2}{4}\right)}{x^l} \Bigg[& {}_1F_1\left(\frac{1-2l-4\epsilon_1}{4}, \frac{1-2l}{2}; \frac{x^2}{2}\right) \\ & + \nu_1 \frac{\Gamma\left(\frac{3+2l-4\epsilon_1}{4}\right)}{\Gamma\left(\frac{3+2l}{2}\right)} \left(\frac{x^2}{2}\right)^{l+\frac{1}{2}} {}_1F_1\left(\frac{3+2l-4\epsilon_1}{4}, \frac{3+2l}{2}; \frac{x^2}{2}\right)\Bigg]. \end{aligned} \tag{3.23}$$

To avoid 1-susy singularities in the domain $x > 0$, we must take $\epsilon_1 \leqslant E_0^{(0)}$ and $\nu_1 \geqslant -\Gamma\left(\frac{1}{2} - l\right)\big/\Gamma\left(\frac{1}{4} - \frac{l}{2} - \epsilon_1\right)$. This restriction on ν_1 changes in the higher order case.

If susy QM is used to create k new levels $\epsilon_i \leqslant E_0^{(0)}$, we will have $\mathrm{Sp}(H_k) = \left\{\epsilon_i, E_n^{(0)} = n + \frac{l}{2} + \frac{3}{4}, i = 1, \ldots, k, n = 0, 1, \ldots\right\}$, i.e., the polynomial algebra (2.1)–(2.4) rules the hsusy partners of the radial oscillator, with natural ladder operators given by

$$L^- = B^\dagger A^- B \qquad L^+ = B^\dagger A^+ B. \tag{3.24}$$

As $A^\pm$ are second-order operators and B, $B^\dagger$ are kth-order ones, then L^- and L^+ are of order $(2k + 2)$ implying that $N(H) = L^+L^-$ is a $(2k + 2)$th-order polynomial in H:

$$N(H) = \left(H - \frac{l}{2} - \frac{3}{4}\right)\left(H + \frac{l}{2} - \frac{1}{4}\right)\prod_{i=1}^{k}(H - \epsilon_i)(H - \epsilon_i - 1). \tag{3.25}$$

The pair of roots $\{\epsilon_i, \epsilon_i + 1\}$ indicates the existence of k physical one-step ladders situated at ϵ_i. There is also one infinite physical ladder with lower end $E_0^{(0)}$. Since $[L^-, L^+] = P_{2k+1}(H)$, it is seen that the hsusy partners of the radial oscillator realize naturally the polynomial algebras (2.1)–(2.4) of odd order.

Up to now we have constructed systems ruled by the polynomial algebra (2.1)–(2.4) through hsusy QM. Now, we will look for the most general systems described by such deformed algebras.

4. Polynomial Heisenberg algebras: general systems

Let us determine the general systems described by the polynomial algebras (2.1)–(2.4). Since for m greater than 3 the calculations are quite involved, we will analyse just the cases with $m = 0, 1, 2, 3$.

4.1. Ladder operators of first order ($m = 0$)

We look for the general Hamiltonian (2.3) and first-order ladder operators

$$L^+ = \frac{1}{\sqrt{2}}\left[-\frac{\mathrm{d}}{\mathrm{d}x} + f(x)\right] \qquad L^- = (L^+)^\dagger \tag{4.1}$$

satisfying equation (2.1). Thus, a system involving V, f, and their derivatives is obtained:

$$f' - 1 = 0 \qquad V' - f = 0. \tag{4.2}$$

Up to coordinate and energy displacements, it turns out that $f(x) = x$ and $V(x) = x^2/2$. This potential has one equidistant infinite ladder starting from the extremal state $\psi_{\mathcal{E}} = \pi^{-1/4}\exp\left(-\frac{x^2}{2}\right)$, which is annihilated by L^- and it is a normalized eigenfunction of H with eigenvalue $\mathcal{E} = 1/2$. Here, the number operator is linear in H, $N(H) = H - \mathcal{E}$, i.e., the general system obeying the polynomial Heisenberg algebra (2.1)–(2.4) with $m = 0$ is the harmonic oscillator.

4.2. *Second-order ladder operators* ($m = 1$)

Let us suppose now that

$$L^+ = \frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + g(x)\frac{\mathrm{d}}{\mathrm{d}x} + h(x)\right] \qquad L^- = (L^+)^\dagger. \tag{4.3}$$

Equation (2.1) leads then to a system of equations for V, g, h, and their derivatives:

$$g' + 1 = 0 \qquad h' + 2V' + g = 0$$

$$h'' + 2V'' + 2gV' + 2h = 0.$$

The general solution (up to coordinate and energy displacements) is given by

$$g(x) = -x \qquad h(x) = \frac{x^2}{4} - \frac{\gamma}{x^2} - \frac{1}{2} \qquad V(x) = \frac{x^2}{8} + \frac{\gamma}{2x^2}. \tag{4.4}$$

The potentials (4.4) have two equidistant energy ladders (not necessarily physical) generated by acting with the powers of L^+ on the two extremal states:

$$\psi_{\mathcal{E}_1} \propto x^{\frac{1}{2}+\sqrt{\gamma+\frac{1}{4}}}\exp\left(-\frac{x^2}{4}\right) \qquad \psi_{\mathcal{E}_2} \propto x^{\frac{1}{2}-\sqrt{\gamma+\frac{1}{4}}}\exp\left(-\frac{x^2}{4}\right). \tag{4.5}$$

Let us recall that $L^-\psi_{\mathcal{E}_i} = 0 = (H - \mathcal{E}_i)\psi_{\mathcal{E}_i}$, where $\mathcal{E}_1 = \frac{1}{2} + \frac{1}{2}\sqrt{\gamma + \frac{1}{4}}$, $\mathcal{E}_2 = \frac{1}{2} - \frac{1}{2}\sqrt{\gamma + \frac{1}{4}}$. Now $N(H)$ is quadratic in H: $N(H) = (H - \mathcal{E}_1)(H - \mathcal{E}_2)$. The potentials (3.18) are recovered by making $\gamma = l(l+1)$, $l \geqslant 0$. Thus, the general systems having second-order ladder operators are described by the radial oscillator potentials (4.4).

4.3. *Ladder operators of third order* ($m = 2$)

Let $L^\pm$ be now third-order ladder operators, factorized by convenience as [12],

$$L^+ = L_1^+L_2^+ \qquad L^- = L_2^-L_1^- \tag{4.6}$$

where $L_1^- = (L_1^+)^\dagger$, $L_2^- = (L_2^+)^\dagger$ and

$$L_1^+ = \frac{1}{\sqrt{2}}\left[-\frac{\mathrm{d}}{\mathrm{d}x} + f(x)\right] \qquad L_2^+ = \frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + g(x)\frac{\mathrm{d}}{\mathrm{d}x} + h(x)\right]. \tag{4.7}$$

It is assumed the existence of an auxiliary Hamiltonian H_a which is intertwined with H as follows:

$$HL_1^+ = L_1^+(H_\mathrm{a} + 1) \qquad H_\mathrm{a}L_2^+ = L_2^+H. \tag{4.8}$$

Thus, we arrive at the following system of equations:

$$-f' + f^2 = 2V - 2\mathcal{E}_3 \tag{4.9}$$

$$V_\mathrm{a} = V + f' - 1 = V + g' \tag{4.10}$$

$$\frac{g''}{2g} - \left(\frac{g'}{2g}\right)^2 - g' + \frac{g^2}{4} + \frac{(\mathcal{E}_1 - \mathcal{E}_2)^2}{g^2} + \mathcal{E}_1 + \mathcal{E}_2 - 2 = 2V. \tag{4.11}$$

By integrating (4.10) (up to a coordinate displacement) we get

$$f(x) = g(x) + x \tag{4.12}$$

and using (4.11), (4.12) in (4.9), we find the following differential equation for $g(x)$,

$$g'' = \frac{g'^2}{2g} + \frac{3}{2}g^3 + 4xg^2 + 2(x^2 + 2\mathcal{E} + 1)g - \frac{2\Delta_1^2}{g} \tag{4.13}$$

which is a Painlevé IV (PIV) equation with $\Delta_1 = \mathcal{E}_1 - \mathcal{E}_2, \mathcal{E} = \mathcal{E}_3 - \frac{1}{2}(\mathcal{E}_1 + \mathcal{E}_2)$ [7, 8, 12, 13, 22, 23]. The potential $V(x)$ can be found by substituting (4.12) into (4.9):

$$V(x) = \frac{x^2}{2} - \frac{g'}{2} + \frac{g^2}{2} + xg + \mathcal{E}_3 - \frac{1}{2}. \tag{4.14}$$

It has three energy ladders, each one with equidistant levels. The extremal states are such that $L^-\psi_{\mathcal{E}_i} = (H - \mathcal{E}_i)\psi_{\mathcal{E}_i} = 0, i = 1, 2, 3$, and are given by

$$\psi_{\mathcal{E}_1} \propto \left(\frac{g'}{2g} - \frac{g}{2} - \frac{\Delta_1}{g} - x\right) \exp\left[\int \left(\frac{g'}{2g} + \frac{g}{2} - \frac{\Delta_1}{g}\right) \mathrm{d}x\right] \tag{4.15a}$$

$$\psi_{\mathcal{E}_2} \propto \left(\frac{g'}{2g} - \frac{g}{2} + \frac{\Delta_1}{g} - x\right) \exp\left[\int \left(\frac{g'}{2g} + \frac{g}{2} + \frac{\Delta_1}{g}\right) \mathrm{d}x\right] \tag{4.15b}$$

$$\psi_{\mathcal{E}_3} \propto \exp\left(-\frac{x^2}{2} - \int g\, \mathrm{d}x\right). \tag{4.15c}$$

Here the generalized number operator is cubic in H:

$$N(H) = (H - \mathcal{E}_1)(H - \mathcal{E}_2)(H - \mathcal{E}_3). \tag{4.16}$$

Hence, we have a recipe for building systems ruled by second-order polynomial algebras (2.1)–(2.4): first find a $g(x)$ obeying the PIV equation (4.13); then calculate the potential, using (4.14), and its three ladders from the extremal states (4.15*a*)–(4.15*c*). In order to test the effectivity of this recipe, let us analyse some systems associated with particular PIV solutions $g(x)$.

(i) *The harmonic oscillator.* Let us consider the following solution of (4.13):

$$g(x) = -x - \alpha(x) \tag{4.17}$$

where $\mathcal{E}_1 = \mathcal{E}_3, \alpha(x) = u'/u$ satisfies the Riccati equation

$$\alpha'(x) + \alpha^2(x) = x^2 - 2\epsilon \tag{4.18}$$

$\epsilon = 2\mathcal{E} + \frac{1}{2} = \mathcal{E}_3 - \mathcal{E}_2 + \frac{1}{2}$, and $u(x)$ is the Schrödinger solution given in (3.15). This $g(x)$ substituted in (4.14) provides

$$V(x) = \frac{x^2}{2} + \mathcal{E}_2 - \frac{1}{2} \tag{4.19}$$

which is the harmonic oscillator potential. The three extremal states (4.15*a*)–(4.15*c*) become

$$\psi_{\mathcal{E}_1} = 0 \qquad \psi_{\mathcal{E}_2} \propto \exp\left(-\frac{x^2}{2}\right) \qquad \psi_{\mathcal{E}_3} \propto u(x). \tag{4.20}$$

We see that the only physical ladder is the one generated from $\psi_{\mathcal{E}_2}$. Here, we have a case where the deformed algebra is reducible in the sense explained at the end of section 2. In fact, it is easy to show that $L^- = a(H - \mathcal{E}_1)$.

(ii) *The 1-susy oscillator partners.* They arise for $g(x)$ taking the form

$$g(x) = -x + \alpha(x) \tag{4.21}$$

where $\alpha = u'/u$ satisfies (4.18), but now $\mathcal{E}_1 = \mathcal{E}_3 + 1$, $\epsilon = 2\mathcal{E} + \frac{3}{2} = \mathcal{E}_3 - \mathcal{E}_2 + \frac{1}{2}$, and $u(x)$ is again the Schrödinger solution (3.15). This $g(x)$ leads to the exactly solvable potentials:

$$V(x) = \frac{x^2}{2} - \alpha'(x) + \mathcal{E}_2 - \frac{1}{2} \tag{4.22}$$

which are the 1-susy partners of the oscillator. The extremal states become

$$\psi_{\mathcal{E}_1} \propto B^\dagger a^\dagger u(x) \qquad \psi_{\mathcal{E}_2} \propto B^\dagger \exp\left(-\frac{x^2}{2}\right) \qquad \psi_{\mathcal{E}_3} \propto \frac{1}{u(x)} \tag{4.23}$$

where $B^\dagger$ is a first-order intertwiner, as in (3.6).

(iii) *The k-susy oscillator partners with $k > 1$.* Recently, a method has been found which allows us to connect the k-independent one-step ladders of the k-susy Hamiltonians $H \equiv H_k$ of section 3.1, to build just a ladder with k steps [24]. The corresponding systems, in principle ruled by the $(2k)$th-order deformed structures (2.1)–(2.4), will be described now by a polynomial Heisenberg algebra of second order, supplying us with more solutions to the PIV equation. The process consists in taking k transformation functions u_i of an unphysical ladder, i.e.,

$$H_0 u_i = \epsilon_i u_i \qquad u_{i+1} \propto a^\dagger u_i \qquad \epsilon_i = \epsilon_1 + i - 1 < \tfrac{1}{2} \qquad i = 1, \ldots, k \tag{4.24}$$

with u_1 being the Schrödinger solution in (3.15). With this choice, the $k - 1$ factorization energies $\epsilon_2 = \epsilon_1 + 1, \ldots, \epsilon_k = \epsilon_1 + k - 1$ appear twice in (3.17), implying that the natural $(2k + 1)$th-order ladder operator of section 3.1 can be written as the product of $(H - \epsilon_2) \cdots (H - \epsilon_k)$ times a third-order ladder operator [24]. This operator leads to the PIV solution we are looking for. Indeed, from the extremal states in the previous two cases it is clear that now

$$\psi_{\mathcal{E}_1} \propto B^\dagger a^\dagger u_k(x) \qquad \psi_{\mathcal{E}_2} \propto B^\dagger \exp\left(-\frac{x^2}{2}\right) \qquad \psi_{\mathcal{E}_3} \propto \frac{W(u_2, \ldots, u_k)}{W(u_1, \ldots, u_k)} \tag{4.25}$$

where $B^\dagger$ is the kth-order intertwining operator of (3.12), $\alpha = u_1'/u_1$ satisfies (4.18) but with $\mathcal{E}_1 = \mathcal{E}_3 + k$, $\epsilon = 2\mathcal{E} + \frac{1}{2} + k = \mathcal{E}_3 - \mathcal{E}_2 + \frac{1}{2}$, and the expression for $\psi_{\mathcal{E}_3}$ will be justified in the appendix. By comparing (4.15c) with (4.25) it turns out that the solution $g(x)$ of the PIV equation (4.13) reads now

$$g(x) = -x - [\ln \psi_{\mathcal{E}_3}]' = -x - [\ln W(u_2, \ldots, u_k)]' + [\ln W(u_1, \ldots, u_k)]' \tag{4.26}$$

and the corresponding potential becomes (see an example in figure 2)

$$V(x) = \frac{x^2}{2} - [\ln W(u_1, \ldots, u_k)]'' + \mathcal{E}_2 - \frac{1}{2}. \tag{4.27}$$

4.4. *Fourth-order ladder operators* ($m = 3$)

Let $L^\pm$ be fourth-order ladder operators factorized as in (4.6) and obeying (4.8) but now

$$L_1^+ = \frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + g_1(x)\frac{\mathrm{d}}{\mathrm{d}x} + h_1(x)\right]. \tag{4.28}$$

An explicit calculation leads to the system of equations:

$$\frac{g_1''}{2g_1} - \left(\frac{g_1'}{2g_1}\right)^2 + g_1' + \frac{g_1^2}{4} + \frac{(\mathcal{E}_3 - \mathcal{E}_4)^2}{g_1^2} + \mathcal{E}_3 + \mathcal{E}_4 = 2V \tag{4.29}$$

$$V_\mathrm{a} = V - g_1' - 1 = V + g' \tag{4.30}$$

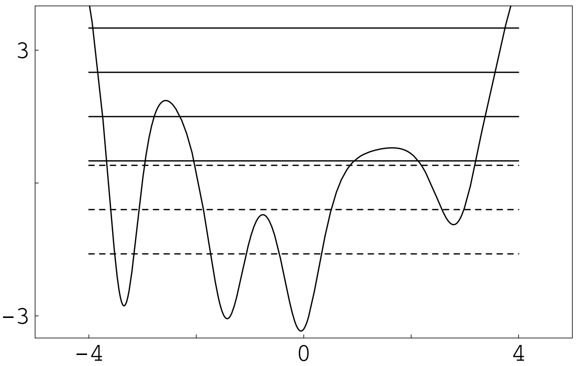

Figure 2. A 3-susy partner potential of the oscillator obtained from (4.27). Its spectrum consists of a three-step ladder (dashed lines at −1.6, −0.6, 0.4) and an infinite ladder (its first four levels at 1/2, 3/2, 5/2, 7/2 are represented by continuous lines).

$$\frac{g''}{2g} - \left(\frac{g'}{2g}\right)^2 - g' + \frac{g^2}{4} + \frac{(\mathcal{E}_1 - \mathcal{E}_2)^2}{g^2} + \mathcal{E}_1 + \mathcal{E}_2 - 2 = 2V. \tag{4.31}$$

From (4.30) we get, up to a coordinate displacement,

$$g_1(x) = -g(x) - x. \tag{4.32}$$

By substituting (4.31), (4.32) into (4.29) one arrives at the differential equation for $g(x)$:

$$g'' = \frac{(2g+x)}{2g(g+x)}(g')^2 - \frac{g}{x(g+x)}g' + R(x,g) \tag{4.33}$$

with

$$\begin{aligned} R(x,g) = {} & [2xg(g+x)]^{-1}[2xg^5 + (5x^2 + 8\mathcal{E} + 4)g^4 + 4x(x^2 + 4\mathcal{E} + 2)g^3 \\ & + \left[x^4 + 4(2\mathcal{E}+1)x^2 + 4\left(\Delta_2^2 - \Delta_1^2\right) - 1\right]g^2 - 4\Delta_1^2 x(2g+x)\big] \end{aligned} \tag{4.34}$$

where $\mathcal{E} = \frac{1}{2}(\mathcal{E}_3 + \mathcal{E}_4) - \frac{1}{2}(\mathcal{E}_1 + \mathcal{E}_2)$, $\Delta_1 = \mathcal{E}_1 - \mathcal{E}_2$ and $\Delta_2 = \mathcal{E}_3 - \mathcal{E}_4$. In order to identify equation (4.33), let us make $g = x/(w-1)$ and then change the variable as $z = x^2$. Thus,

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2}{z^2}\left(\frac{aw^2+b}{w}\right) + \frac{cw}{z} + \frac{dw(w+1)}{w-1} \tag{4.35}$$

which is a Painlevé V equation (PV) with $a = \frac{\Delta_1^2}{2}$, $b = -\frac{\Delta_2^2}{2}$, $c = -\mathcal{E} - \frac{1}{2}$, $d = -\frac{1}{8}$ [8, 22]. The spectrum contains four independent equidistant ladders starting from the extremal states,

$$\begin{aligned} \psi_{\mathcal{E}_1} &\propto \left[\frac{g_1}{2}\left(\frac{g'}{2g} - \frac{g_1'}{2g_1} + \frac{g+g_1}{2} - \frac{\Delta_1}{g}\right) + \mathcal{E} - \frac{\Delta_1}{2}\right]\exp\left[\int\left(\frac{g'}{2g} + \frac{g}{2} - \frac{\Delta_1}{g}\right)\mathrm{d}x\right] \\ \psi_{\mathcal{E}_2} &\propto \left[\frac{g_1}{2}\left(\frac{g'}{2g} - \frac{g_1'}{2g_1} + \frac{g+g_1}{2} + \frac{\Delta_1}{g}\right) + \mathcal{E} + \frac{\Delta_1}{2}\right]\exp\left[\int\left(\frac{g'}{2g} + \frac{g}{2} + \frac{\Delta_1}{g}\right)\mathrm{d}x\right] \\ \psi_{\mathcal{E}_3} &\propto \exp\left[\int\left(\frac{g_1'}{2g_1} + \frac{g_1}{2} - \frac{\Delta_2}{g_1}\right)\mathrm{d}x\right] \\ \psi_{\mathcal{E}_4} &\propto \exp\left[\int\left(\frac{g_1'}{2g_1} + \frac{g_1}{2} + \frac{\Delta_2}{g_1}\right)\mathrm{d}x\right]. \end{aligned} \tag{4.36}$$

The number operator is a fourth-order polynomial in H:

$$N(H) = (H - \mathcal{E}_1)(H - \mathcal{E}_2)(H - \mathcal{E}_3)(H - \mathcal{E}_4). \tag{4.37}$$

Hence, given a solution $w(z)$ of the PV equation (4.35) (or $g(x)$ for (4.33)), a system characterized by the third-order polynomial algebra (2.1)–(2.4) with potential (4.31) and extremal states (4.36) is straightforwardly constructed. In order to test our recipe, let us discuss some explicit examples.

(*a*) *The radial oscillator.* Let us consider the following solution of equation (4.33),

$$g(x) = -\frac{x}{2} - \frac{l}{x} - \alpha(x) \tag{4.38}$$

where $\mathcal{E}_1 = \mathcal{E}_3, l = 2\mathcal{E} + \frac{1}{2} = \mathcal{E}_4 - \mathcal{E}_2 + \frac{1}{2}$ and $\alpha(x)$ is a solution of the Riccati equation

$$\alpha'(x) + \alpha^2(x) = 2\left[\frac{x^2}{8} + \frac{l(l+1)}{2x^2} - \epsilon\right] \tag{4.39}$$

where $\epsilon = \left(\Delta_1^2 - \Delta_2^2\right)/(4\mathcal{E}) = \mathcal{E}_3 - (\mathcal{E}_2 + \mathcal{E}_4)/2$. The function $\alpha(x)$ can be written as $\alpha(x) = u'/u$, with u given by (3.23). The corresponding potentials (4.31) become

$$V(x) = \frac{x^2}{8} + \frac{l(l-1)}{2x^2} + \frac{\mathcal{E}_4 + \mathcal{E}_2 - 1}{2} \tag{4.40}$$

i.e., the radial oscillator. Its corresponding extremal states (4.36) read

$$\psi_{\mathcal{E}_1} = 0 \qquad \psi_{\mathcal{E}_2} \propto x^{-(l-1)} \exp\left(-\frac{x^2}{4}\right) \tag{4.41a}$$

$$\psi_{\mathcal{E}_3} \propto \left[u' - \left(\frac{x}{2} - \frac{l}{x}\right)u\right] \qquad \psi_{\mathcal{E}_4} \propto x^l \exp\left(-\frac{x^2}{4}\right). \tag{4.41b}$$

The fact that the radial oscillator is a system described by a Lie algebra and, at the same time, by a deformed algebra, tells us that the latter must be reducible in the same way as mentioned in example (i).

(*b*) *The 1-susy partners of the radial oscillator.* If we take now

$$g(x) = -\frac{x}{2} - \frac{(l+1)}{x} + \alpha(x) \tag{4.42}$$

it turns out that $\alpha = u'/u$ satisfies again the Riccati equation (4.39), but now $\mathcal{E}_1 = \mathcal{E}_3 + 1, l = 2\mathcal{E} + \frac{1}{2} = \mathcal{E}_4 - \mathcal{E}_2 - \frac{1}{2}, \epsilon = \left(\Delta_1^2 - \Delta_2^2\right)/[4(\mathcal{E}+1)] = \mathcal{E}_3 - (\mathcal{E}_2 + \mathcal{E}_4 - 1)/2$, and u given in (3.23). The potentials (4.31) now become

$$V(x) = \frac{x^2}{8} + \frac{l(l+1)}{2x^2} + \frac{\mathcal{E}_2 + \mathcal{E}_4 - 1}{2} - \alpha'(x) \tag{4.43}$$

i.e., the 1-susy partners of the radial oscillator. The four extremal states read

$$\psi_{\mathcal{E}_1} \propto B^\dagger A^+ u \qquad \psi_{\mathcal{E}_2} \propto B^\dagger \left(x^{-l} \exp\left(-\frac{x^2}{4}\right)\right) \tag{4.44a}$$

$$\psi_{\mathcal{E}_3} \propto \frac{1}{u} \qquad \psi_{\mathcal{E}_4} \propto B^\dagger \left(x^{l+1} \exp\left(-\frac{x^2}{4}\right)\right) \tag{4.44b}$$

where $B^\dagger$ is the first-order intertwiner and A^+ is the second-order ladder operator in (3.20).

(*c*) *The k-susy radial oscillator partners with* $k > 1$. As in the example (iii) of section 4.3, a reduction process allows us to assemble the one-step ladders of certain k-susy partner Hamiltonians of the radial oscillator. Then, the natural $(2k+1)$th-order polynomial algebra ruling $H \equiv H_k$ reduces to a third-order one, leading then to solutions of the PV equation.

Indeed, let us take the k transformation functions once again as the steps of an unphysical ladder of the radial oscillator potential $V_0(x)$ in (3.18), i.e.,

$$H_0 u_i = \epsilon_i u_i \qquad u_{i+1} \propto A^+ u_i \qquad \epsilon_i = \epsilon_1 + i - 1 < \frac{l}{2} + \frac{3}{4} \qquad i = 1, \ldots, k. \tag{4.45}$$

From the previous example (b), one immediately finds the four extremal states associated with the reduced third-order polynomial algebra,

$$\psi_{\mathcal{E}_1} \propto B^\dagger A^+ u_k \qquad \psi_{\mathcal{E}_2} \propto B^\dagger \left(x^{-l} \exp\left(-\frac{x^2}{4}\right)\right) \qquad \psi_{\mathcal{E}_3} \propto \frac{W(u_2, \ldots, u_k)}{W(u_1, \ldots, u_k)} \tag{4.46a}$$

$$\psi_{\mathcal{E}_4} \propto B^\dagger \left(x^{l+1} \exp\left(-\frac{x^2}{4}\right)\right) \propto \frac{W\big(u_1, \ldots, u_k, x^{l+1} \exp\left(-\frac{x^2}{4}\right)\big)}{W(u_1, \ldots, u_k)} \tag{4.46b}$$

where now $B^\dagger$ is the kth-order intertwining operator of (3.12), $\alpha = u_1'/u_1$ satisfies again (4.39) but with $\epsilon = \left(\Delta_1^2 - \Delta_2^2\right)/[4(\mathcal{E} + k)] - (k-1)/2 = \mathcal{E}_3 - (\mathcal{E}_2 + \mathcal{E}_4 - 1)/2$, $\mathcal{E}_1 = \mathcal{E}_3 + k$, $l = 2\mathcal{E} - 1/2 + k = \mathcal{E}_4 - \mathcal{E}_2 - \frac{1}{2}$, and u_1 is the solution of (3.23). The last formula in (4.46a) will be discussed in the appendix. By comparing the expressions for $\psi_{\mathcal{E}_3}$ and $\psi_{\mathcal{E}_4}$ in (4.36) and (4.46a), (4.46b), we immediately obtain

$$g_1(x) = \frac{2\Delta_2}{\left[\ln\left(\frac{\psi_{\mathcal{E}_4}}{\psi_{\mathcal{E}_3}}\right)\right]'} = \frac{2\Delta_2 W(u_2, \ldots, u_k) W\big(u_1, \ldots, u_k, x^{l+1} \exp\left(-\frac{x^2}{4}\right)\big)}{W\big(W(u_2, \ldots, u_k), W\big(u_1, \ldots, u_k, x^{l+1} \exp\left(-\frac{x^2}{4}\right)\big)\big)}. \tag{4.47}$$

Therefore,

$$g(x) = -x - \frac{2\Delta_2}{\left[\ln\left(\frac{\psi_{\mathcal{E}_4}}{\psi_{\mathcal{E}_3}}\right)\right]'} = -x - \frac{2\Delta_2 W(u_2, \ldots, u_k) W\big(u_1, \ldots, u_k, x^{l+1} \exp\left(-\frac{x^2}{4}\right)\big)}{W\big(W(u_2, \ldots, u_k), W\big(u_1, \ldots, u_k, x^{l+1} \exp\left(-\frac{x^2}{4}\right)\big)\big)} \tag{4.48}$$

which is a solution of (4.33) and it is directly related to the corresponding PV solution through $w(z) = 1 + \sqrt{z}/g(\sqrt{z})$. Finally, the potentials (4.31) are

$$V(x) = \frac{x^2}{8} + \frac{l(l+1)}{2x^2} - [\ln W(u_1, \ldots, u_k)]'' + \frac{\mathcal{E}_2 + \mathcal{E}_4 - 1}{2}. \tag{4.49}$$

5. Conclusions

In this paper, we have presented a short overview of the polynomial Heisenberg algebras and explained how the methods of hsusy QM can be useful in this subject. So, we have shown that the higher order supersymmetric partners of the harmonic and radial oscillators provide the simplest non-trivial realizations of those deformed structures [11, 24]. We have analysed as well the general systems ruled by the polynomial Heisenberg algebras when the differential ladder operators are of order one, two, three and four ($m = 0, 1, 2, 3$ respectively), and we have proved that the corresponding potentials involve Painlevé transcendents of types IV and V in the last two cases (see also [7, 8, 13]). Although parts of the results included here were known in different contexts, we thought it was the right time to join them together in a self-contained way and with a unified viewpoint. We must remark, however, a number of original results. Let us mention, for instance, the treatment of the whole section 4.4 devoted to the Painlevé V equation. Also, we have explored and generalized a reduction process using the k-susy QM applied to the harmonic [24] and radial oscillators to construct a class of exact solutions to PIV and PV equations. The importance of our technique can be appreciated by comparing it with more involved methods used to get PIV and PV solutions (see, for instance, [23]). The existence of other kinds of solutions, although worth studying, is outside the scope of this paper (see, however, [25]).

Acknowledgments

The authors acknowledge the support of CONACYT (Mexico), project no 40888-F, and the Spanish MCYT (projects BFM2002-02000 and BFM2002-03773) and Junta de Castilla y León (VA085/02). JMC also thanks the ESFM-IPN, ESCOM-IPN and CINVESTAV for support.

Appendix

This appendix is devoted to justify the formulae which have been used in (4.25) and (4.46*a*). As a particular case of (3.1), let us consider a 2-susy transformation $B^\dagger$ intertwining H_0 and H_2 as $H_2 B^\dagger = B^\dagger H_0$, generated by the two transformation functions u_i such that $H_0 u_i = \epsilon_i u_i, i = 1, 2$. We can express $B^\dagger$ in terms of u_i in two ways, either by means of Wronskians or through two consecutive 1-susy transformations,

$$B^\dagger \psi = \frac{W(u_1, u_2, \psi)}{W(u_1, u_2)} \tag{A.1}$$

$$\equiv \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\left(u_2^t\right)'}{u_2^t}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{u_1'}{u_1}\right)\psi \tag{A.2}$$

$$\equiv \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\left(u_1^t\right)'}{u_1^t}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{u_2'}{u_2}\right)\psi \tag{A.3}$$

where u_1^t is the result of the 1-susy transformation onto u_1 when u_2 is employed, and similarly for u_2^t, namely $u_1^t \propto W(u_1, u_2)/u_2$ and $u_2^t \propto W(u_1, u_2)/u_1$. From any of the equations (A.1) or (A.2), (A.3), we can check that $B^\dagger u_i = 0, i = 1, 2$. By taking the adjoint of $B^\dagger$ we get the operator B realizing the intertwining in the opposite way, $H_0 B = B H_2$. If we choose equations (A.2), (A.3), the following expressions for B are obtained:

$$B \equiv \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{u_1'}{u_1}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\left(u_2^t\right)'}{u_2^t}\right) \tag{A.4}$$

$$\equiv \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{u_2'}{u_2}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\left(u_1^t\right)'}{u_1^t}\right). \tag{A.5}$$

Now, from (A.4), (A.5) we find easily the eigenfunctions $H_2 \widetilde{u}_i = \epsilon_i \widetilde{u}_i$ which are annihilated by B. They are given by

$$\widetilde{u}_1 = \frac{1}{u_1^t} \propto \frac{u_2}{W(u_1, u_2)} \qquad \widetilde{u}_2 = \frac{1}{u_2^t} \propto \frac{u_1}{W(u_1, u_2)}. \tag{A.6}$$

If we repeat exactly the same arguments for an nth-order susy transformation $H_n B^\dagger = B^\dagger H_0$, generated by n transformation functions u_i such that $H_0 u_i = \epsilon_i u_i, i = 1, \ldots, n$ and $B^\dagger u_i = 0$, then the adjoint operator B performing the intertwining in the opposite direction is characterized by n eigenfunctions $H_n \widetilde{u}_i = \epsilon_i \widetilde{u}_i, i = 1, \ldots, n$ such that $B\widetilde{u}_i = 0$. These eigenfunctions are given by

$$\begin{aligned} \widetilde{u}_1 &= \frac{1}{u_1^t} = \frac{W(u_2, \ldots, u_n)}{W(u_1, \ldots, u_n)} \\ &\vdots \\ \widetilde{u}_n &= \frac{1}{u_n^t} = \frac{W(u_1, \ldots, u_{n-1})}{W(u_1, \ldots, u_n)} \end{aligned} \tag{A.7}$$

where u_i^t is the result of an $(n-1)$th-order susy transformation onto u_i when the transformation functions are the $n-1$ remaining u_j, $j = 1, \ldots, i-1, i+1, \ldots, n$. These are just the expressions appearing in (4.25) and (4.46*a*).

References

[1] Higgs P W 1979 *J. Phys. A: Math. Gen.* **12** 309
[2] Bonatsos D, Daskaloyannis C and Kokkotas K 1994 *Phys. Rev.* A **50** 3700
[3] Roček M 1991 *Phys. Lett.* B **255** 554
Daskaloyannis C 1991 *J. Phys. A: Math. Gen.* **24** L789
Beckers J, Brihaye Y and Debergh N 1999 *J. Phys. A: Math. Gen.* **32** 2791
Sunilkumar V, Bambah B A, Jagannathan R, Panigrahi P K and Srinivasan V 2000 *J. Opt.* B **2** 126
[4] Dutt R, Gangopadhyaya A, Rasinariu C and Sukhatme U P 1999 *Phys. Rev.* A **60** 3482
[5] Fernández D J 1984 *MSc Thesis* México DF, Cinvestav
[6] Dubov S Y, Eleonsky V M and Kulagin N E 1992 *Sov. Phys.–JETP* **75** 446
[7] Veselov A P and Shabat A B 1993 *Funct. Anal. Appl.* **27** 81
[8] Adler V E 1994 *Physica* D **73** 335
[9] Sukhatme U P, Rasinariu C and Khare A 1997 *Phys. Lett.* A **234** 401
[10] Aizawa N and Sato H T 1997 *Prog. Theor. Phys.* **98** 707
Cannata F, Junker G and Trost J 1998 *Particles, Fields and Gravitation* (*AIP Conf. Proc.* vol 453) ed J Rembielinski (New York: AIP) p 209
Arik M, Atakishiyev N M and Wolf K B 1999 *J. Phys. A: Math. Gen.* **32** L371
[11] Fernández D J and Hussin V 1999 *J. Phys. A: Math. Gen.* **32** 3603
[12] Andrianov A A, Cannata F, Ioffe M and Nishnianidze D 2000 *Phys. Lett.* A **266** 341
[13] Willox R and Hietarinta J 2003 *J. Phys. A: Math. Gen.* **36** 10615
[14] Quesne C and Vansteenkiste N 1999 *Helv. Phys. Acta* **72** 71
[15] Andrianov A A, Ioffe M V and Spiridonov V 1993 *Phys. Lett.* A **174** 273
Andrianov A A, Ioffe M V, Cannata F and Dedonder J P 1995 *Int. J. Mod. Phys.* A **10** 2683
Fernández D J 1997 *Int. J. Mod. Phys.* A **12** 171
Fernández D J, Glasser M L and Nieto L M 1998 *Phys. Lett.* A **240** 15
Aoyama H, Nakayama N, Sato M and Tanaka T 2001 *Phys. Lett.* B **521** 400
Andrianov A A and Sokolov A V 2003 *Nucl. Phys.* B **660** 25
Cannata F, Ioffe M and Nishnianidze D 2003 *Phys. Lett.* A **310** 344
[16] Bagrov V G and Samsonov B F 1997 *Phys. Part. Nucl.* **28** 374
[17] Fernández D J, Hussin V and Mielnik B 1998 *Phys. Lett.* A **244** 309
Mielnik B, Nieto L M and Rosas-Ortiz O 2000 *Phys. Lett.* A **269** 70
[18] Rosas-Ortiz J O 1998 *J. Phys. A: Math. Gen.* **31** L507
Rosas-Ortiz J O 1998 *J. Phys. A: Math. Gen.* **31** 10163
Cariñena J F, Ramos A and Fernández D J 2001 *Ann. Phys., NY* **292** 42
[19] Carballo J M 2001 *MSc Thesis* México DF, ESFM-IPN
[20] Mielnik B 1984 *J. Math. Phys.* **25** 3387
Fernández D J, Hussin V and Nieto L M 1994 *J. Phys. A: Math. Gen.* **27** 3547
Fernández D J, Nieto L M and Rosas-Ortiz O 1995 *J. Phys. A: Math. Gen.* **28** 2693
[21] Fernández D J, Negro J and del Olmo M A 1996 *Ann. Phys., NY* **252** 386
Mota R D, Granados V D, Queijeiro A, García J and Guzmán L 2003 *J. Phys. A: Math. Gen.* **36** 4849
[22] Iwasaki K, Kimura H, Shimomura S and Yoshida M 1991 *From Gauss to Painlevé* (Braunschweig: Friedrich Vieweg & Sohn)
[23] Bassom A P, Clarkson P A and Hicks A C 1995 *Stud. Appl. Math.* **95** 1
[24] Fernández D J, Negro J and Nieto L M 2004 *Phys. Lett.* A (at press)
[25] Winternitz P and Levi D (ed) 1992 *Painlevé Transcendents, Their Asymptotics and Physical Applications, (NATO ASI Series B)* (New York: Plenum)
Kapaev A A 1998 *Preprint* solv-int/9805011